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ON A LATTICE POINT PROBLEM OF L. MOSER. I

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We prove the following conjecture of L. Moser: Any convex region of area n can be placed so as to cover $\geqq n+f(n)$ lattice points, where $f(n)\to\infty$.

1. Introduction

In 1959 Leo Moser [6] raised the following problem.

"It is well known that any region A of area x can be placed so as to cover $\geqq x$ lattice points. Assume now that the region is convex. Can it be placed so as to cover $\geqq x+f(x)$ lattice points, where $f(x)\to\infty$ as $x\to\infty$?

There is no analogue for translates of A: let Q be a square parallel to the coordinate axis with $\text{area}(Q)=n^2-1$, n is an integer; then for all $\mathbf{x}\in\mathbf{R}^2$, $\text{card}(Q+\mathbf{x})\cap\mathbf{Z}^2\leqq n^2$." (Card stands for cardinality.) This formulation of the question is cited from W. Moser's problem collection [7].

Our object is to give an affirmative answer to this question.

Theorem 1.1. *There is a universal function $f(x)$, $f(x)\geqq x^{1/9}$ for $x\geqq c_0$ (c_0 is a positive absolute constant) such that any convex region A can be placed on the plane so as to cover $\geqq x+f(x)$ lattice points, where $x=\text{area}(A)$.*

Remarks. The same proof gives that one can also place A so as to cover $\leqq x-f(x)$ lattice points, where $x=\text{area}(A)$.

The particular case $A=$*circular disc* was earlier solved by M. M. Skriganov [9] with $f(x)=x^{1/6-\varepsilon}$ (note that Skriganov dealt with arbitrary lattices in $\mathbf{R}^2$, not just the usual square lattice $\mathbf{Z}^2$). He applied this "irregularity" type result to study the spectrum of the two-dimensional Schrödinger operator with periodic potential function.

In the proof we need a particular case of the following very deep theorem of W. M. Schmidt [8] in Diophantine Approximation: Suppose $y_1, y_2, \dots, y_h$ are real algebraic numbers such that $1, y_1, \dots, y_h$ are linearly independent over the rationals, and suppose $c>1$. There are only finitely many positive integers q with

$$q^c\cdot\|y_1\cdot q\|\cdot\|y_2\cdot q\|\dots\|y_h\cdot q\| < 1 \tag{1.1}$$

($\|\xi\|$ stands for the distance from a real number ξ to the nearest integer).

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Unfortunately, one can at present not give an upper bound $B=B(y_1, y_2, \ldots, y_h, c)$ for solutions q of (1.1). Hence, Schmidt's theorem is "ineffective". This is the reason that our threshold constant c_0 is also "ineffective".

A straightforward modification of the proof yields the sharper lower bound $f(x) \geqq x^{1/8-\varepsilon}$ for $x \geqq c_0(\varepsilon)$. We suspect that the true order of magnitude of $f(x)$ is about $x^{1/4}$.

At present we are unable to generalize Theorem 1.1 in higher dimensions.

2. Deduction of Theorem 1.1 from Theorem 2.1

For any compact and convex region $B \subset \mathbf{R}^2$, let $\Gamma(B)$, $r(B)$, $d(B)$ and $l(B)$ denote the boundary arc of B, the radius of the largest inscribed circle of B, the diameter of B and the perimeter of B, respectively. Let μ denote the usual area function (i.e., the two-dimensional Lebesgue measure).

In Theorem 1.1 we may assume that $r(A) \geqq \frac{1}{9}$. Indeed, in the opposite case, using the following rough estimate (we shall prove it later)

$$d(A) \cdot r(A) \geqq \frac{2}{9} \mu(A), \tag{2.1}$$

we obtain that $d(A) \geqq 2\mu(A)$, and A can be placed so as to cover $\geqq d(A) \geqq 2\mu(A)$ lattice points.

The verification of (2.1) goes as follows. By a classical theorem of W. Blaschke [3] (see also L. Fejes Tóth [4]) there is a triangle $T \subset A$ such that $\mu(T) \geqq \mu(A) \cdot \frac{3}{2\pi} \sin\left(\frac{2\pi}{3}\right)$ (equality holds only for the ellipse). Thus we have $3d(A) \cdot r(A) \geqq l(T) \cdot r(T) = 2\mu(T) \geqq 2\mu(A) \cdot \frac{3}{2\pi} \sin\left(\frac{2\pi}{3}\right) > \frac{2}{3} \mu(A)$, and (2.1) follows.

For any bounded set $S \subset \mathbf{R}^2$ and $\mathbf{x} \in \mathbf{R}^2$, let

$$g(S, \mathbf{x}) = \operatorname{card}(S + \mathbf{x}) \cap \mathbf{Z}^2, \tag{2.2}$$

i.e., the number of lattice points covered by the translate $S+\mathbf{x}$ of S. For any positive real number $\varepsilon \in \left(0, \frac{1}{2}\right]$, let

$$g(S, \mathbf{x}, \varepsilon) = \frac{1}{4\varepsilon^2} \int_{[-\varepsilon, \varepsilon)^2} g(S, \mathbf{x} + \mathbf{y})\, d\mathbf{y}. \tag{2.3}$$

Note that $g\left(S, \mathbf{x}, \frac{1}{2}\right) = \mu(S)$ if S is Lebesgue-measurable and $\lim_{\varepsilon \to 0} g(S, \mathbf{x}, \varepsilon) = g(S, \mathbf{x})$ if there is no lattice point on the boundary of $S + \mathbf{x}$.

Given any angle $\tau \in [0, 2\pi)$, let τS denote the rotated image of $S \subset \mathbf{R}^2$. Let $\mathcal{U}^2 = [0, 1)^2$. We state

Theorem 2.1. *There exist an "ineffective" absolute constant c_0 and an "effective" absolute constant $c_1 > 0$ such that for any convex region A with $\mu(A) \geqq c_0$ and*

$r(A) \geqq \frac{1}{9}$, *we have with* $\varepsilon_0 = (d(A))^{-(1/100)}$,

$$\frac{1}{2\pi}\int_0^{2\pi}\Big(\int_{\mathcal{U}^2}(g(\tau A, \mathbf{y}, \varepsilon_0)-\mu(A))^2\,d\mathbf{y}\Big)\,d\tau \geqq c_1\cdot(d(A))^{97/100}.$$

First we derive Theorem 1.1 from Theorem 2.1. Suppose that $\mu(A)\geqq c_0$ and $r(A)\geqq\frac{1}{9}$. By Theorem 2.1 there exists $\tau^*\in[0, 2\pi)$ such that

$$\int_{\mathcal{U}^2}(g(\tau^* A, \mathbf{y}, \varepsilon_0)-\mu(A))^2\,d\mathbf{y} \geqq c_1\cdot(\mathrm{d}(A))^{97/100}. \tag{2.4}$$

Now the simple idea is as follows. We first show that the $\mathcal{U}^2$-periodic function $g(\tau^* A, \mathbf{y}, \varepsilon_0)$, $\mathbf{y}\in\mathbf{R}^2$ varies rather slowly, i.e. the ratio

$$\frac{g(\tau^* A, \mathbf{y}, \varepsilon_0)-g(\tau^* A, \mathbf{z}, \varepsilon_0)}{|\mathbf{y}-\mathbf{z}|}$$

is not too large (note that this is the very reason for studying the auxiliary function $g(\tau^* A, \mathbf{y}, \varepsilon_0)$ instead of the original function $g(\tau^* A, \mathbf{y})$). Using this property of $g(\tau^* A, \mathbf{y}, \varepsilon_0)$ we will be able to obtain a nontrivial one-sided bound from the L^2-norm estimate (2.4).

For convencience write $A^*=\tau^* A$ and $\mathcal{U}(\mathbf{n}, \varepsilon)=[-\varepsilon, \varepsilon)^2+\mathbf{n}$, $\mathbf{n}\in\mathbf{Z}^2$. Our starting point is the following obvious identity

$$g(A^*, \mathbf{y}, \varepsilon_0) = \frac{1}{4\varepsilon_0^2}\sum_{\mathbf{n}\in\mathbf{Z}^2}\mu((A^*+\mathbf{y})\cap\mathcal{U}(\mathbf{n}, \varepsilon_0)). \tag{2.5}$$

From (2.5) it follows that for any pair $\mathbf{y}, \mathbf{z}\in\mathbf{R}^2$,

(2.6)

$$g(A^*, \mathbf{y}, \varepsilon_0)-g(A^*, \mathbf{z}, \varepsilon_0) = \frac{1}{4\varepsilon_0^2}\sum{}^*(\mu((A^*+\mathbf{y})\cap\mathcal{U}(\mathbf{n}, \varepsilon_0))-\mu((A^*+\mathbf{z})\cap\mathcal{U}(\mathbf{n}, \varepsilon_0)))$$

where the summation $\sum^*$ is taken over all $\mathbf{n}\in\mathbf{Z}^2$ such that

$$\mathcal{U}(\mathbf{n}, \varepsilon_0)\cap\big(\Gamma(A^*+y)\cup\Gamma(A^*+\mathbf{z})\big) \neq \emptyset. \tag{2.7}$$

We have for any $\mathbf{y}\in\mathbf{R}^2$,

$$\operatorname{card}\{\mathbf{n}\in\mathbf{Z}^2\colon \mathcal{U}(\mathbf{n}, \varepsilon_0)\cap\Gamma(A^*+\mathbf{y}) \neq \emptyset\} < 4\big(d(A^*)+1\big) = 4\big(d(A)+1\big). \tag{2.8}$$

Moreover, for any pair $\mathbf{y}, \mathbf{z}\in\mathbf{R}^2$,

$$\big|\mu\big((A^*+\mathbf{y})\cap\mathcal{U}(\mathbf{n}, \varepsilon_0)\big)-\mu\big((A^*+\mathbf{z})\cap\mathcal{U}(\mathbf{n}, \varepsilon_0)\big)\big| \leqq 4\varepsilon_0\cdot|\mathbf{y}-\mathbf{z}|, \tag{2.9}$$

where $|\mathbf{y}-\mathbf{z}|=((y_1-z_1)^2+(y_2-z_2)^2)^{1/2}$ stands for the Euclidean distance of $\mathbf{y}$ and $\mathbf{z}$. From (2.6)—(2.9) it follows that

$$|g(A^*, \mathbf{y}, \varepsilon_0)-g(A^*, \mathbf{z}, \varepsilon_0)| < \frac{8\big(d(A)+1\big)|\mathbf{y}-\mathbf{z}|}{\varepsilon_0}. \tag{2.10}$$

By (2.5) we have that $g(A^*, \mathbf{y}, \varepsilon_0)$, $\mathbf{y}\in\mathbf{R}^2$ is $\mathcal{U}^2$-periodic and

$$\int_{\mathcal{U}^2} g(A^*, \mathbf{y}, \varepsilon_0)\,d\mathbf{y} = \mu(A). \tag{2.11}$$

Let

$$h(\mathbf{y}) = g(A^*, \mathbf{y}, \varepsilon_0) - \mu(A), \quad \mathbf{y}\in\mathcal{U}^2. \tag{2.12}$$

Note that the function $h(\mathbf{y})$, $\mathbf{y}\in\mathcal{U}^2$ is continuous. Let

$$M = \max_{\mathbf{y}\in\mathcal{U}^2} h(\mathbf{y}) \quad \text{and} \quad -m = \min_{\mathbf{y}\in\mathcal{U}^2} h(\mathbf{y}).$$

Observe that

$$\max_{\mathbf{y}\in\mathcal{U}^2} \operatorname{card}(A^*+\mathbf{y})\cap\mathbf{Z}^2 = \max_{\mathbf{y}\in\mathcal{U}^2} g(A^*, \mathbf{y}) \geqq \max_{\mathbf{y}\in\mathcal{U}^2} g(A^*, \mathbf{y}, \varepsilon_0) = \mu(A)+M. \tag{2.13}$$

By (2.11) and (2.12), $M\geqq 0\geqq -m$, and by (2.4), $M>0>-m$. Let $m_k=2^{-(k-1)}\cdot m$ ($k=1, 2, 3, \ldots$) (so $m_1=m$) and $\mu_k=\mu\left\{\mathbf{y}\in\mathcal{U}^2:\ -\frac{m_k}{2}>h(\mathbf{y})\geqq -m_k\right\}$. For later purpose we mention here the following consequence of (2.11),

$$M \geqq \int_{\substack{\mathbf{y}:\\ h(\mathbf{y})>0}} h(\mathbf{y})\,d\mathbf{y} = -\int_{\substack{\mathbf{y}:\\ h(\mathbf{y})<0}} h(\mathbf{y})\,d\mathbf{y} \geqq \sum_{k=1}^{\infty} \mu_k\cdot\frac{m_k}{2}. \tag{2.14}$$

By (2.4) we have

$$M^2+\sum_{k=1}^{\infty}\mu_k\cdot m_k^2 \geqq \int_{\mathcal{U}^2} h^2(\mathbf{y})\,d\mathbf{y} \geqq c_1\cdot(d(A))^{97/100}. \tag{2.15}$$

If $M^2\geqq\frac{c_1}{2}(d(A))^{97/100}$, then we are immediately done. Indeed, by (2.13) we have

$$\max_{\mathbf{y}\in\mathcal{U}^2} \operatorname{card}(A^*+\mathbf{y})\cap\mathbf{Z}^2 \geqq \mu(A)+\left(\frac{c_1}{2}(d(A))^{97/100}\right)^{1/2} \geqq \mu(A)+c_2\cdot(\mu(A))^{97/400},$$

and Theorem 1.1 is "over-fulfilled" (throughout $c_1, c_2, c_3, \ldots$ are "effective" positive absolute constants). Thus we may assume that (see (2.15))

$$\sum_{k=1}^{\infty}\mu_k\cdot m_k^2 \geqq \frac{c_1}{2}\cdot(d(A))^{97/100}.$$

One can therefore find constants $\alpha_k\geqq 0$, $k=1, 2, 3, \ldots$ such that $\sum_{k=1}^{\infty}\alpha_k=1$ and

$$\mu_k\cdot m_k^2 \geqq \alpha_k\frac{c_1}{2}(d(A))^{97/100} \quad \text{for all} \quad k\geqq 1. \tag{2.16}$$

From (2.10) it follows that the set $\{\mathbf{y}\in\mathbf{R}^2:\ -m_k/2>h(\mathbf{y})\geqq -m_k\}$ contains a circular disc $C(\varrho, \mathbf{c})$ of radius

$$\varrho = \varrho_k = \frac{\varepsilon_0\cdot m_k}{32(d(A)+1)}$$

and centre $\mathbf{c}=\mathbf{c}_k\in\mathbf{R}^2$ such that $h(\mathbf{c})=-\frac{3}{4}m_k$. Since $m_k\leqq m<l(A)<4d(A)$, we have that $\varrho_k<\frac{1}{2}$ and therefore

$$\mu_k=\mu\left\{\mathbf{y}\in\mathscr{U}^2:\ -\frac{m_k}{2}>h(\mathbf{y})\geqq -m_k\right\}\geqq\varrho_k^2\cdot\pi\geqq c_3\cdot\frac{\varepsilon_0^2\cdot m_k^2}{(d(A))^2}. \tag{2.17}$$

By (2.16) and (2.17) we obtain that $\mu_k\cdot m_k\geqq\max\{a_k,b_k\}$ where

$$a_k=\frac{\alpha_k\cdot\frac{c_1}{2}\cdot(d(A))^{97/100}}{m_k}\quad\text{and}\quad b_k=c_3\cdot\frac{\varepsilon_0^2\cdot m_k^3}{(d(A))^2}.$$

Hence

$$\mu_k\cdot m_k\geqq\max\{a_k,b_k\}\geqq(a_k^3b_k)^{1/4}=c_4\cdot\varepsilon_0^{1/2}\cdot\alpha_k^{3/4}\cdot(d(A))^{91/400}. \tag{2.18}$$

By (2.13), (2.14) and (2.18) we have

$$\max_{\mathbf{y}\in\mathscr{U}^2}\left(\operatorname{card}(A^*+\mathbf{y})\cap\mathbf{Z}^2-\mu(A)\right)\geqq M\geqq\frac{1}{2}\sum_{k=1}^{\infty}\mu_k\cdot m_k\geqq$$

$$\geqq\frac{1}{2}c_4\cdot\varepsilon_0^{1/2}\cdot(d(A))^{91/400}\Big(\sum_{k=1}^{\infty}\alpha_k^{3/4}\Big)\geqq\frac{1}{2}c_4\cdot\varepsilon_0^{1/2}\cdot(d(A))^{91/400}\cdot\Big(\sum_{k=1}^{\infty}\alpha_k\Big)=$$

$$=\frac{1}{2}c_4\cdot\varepsilon_0^{1/2}\cdot(d(A))^{91/400}.$$

Since $\varepsilon_0=(d(A))^{-1/100}$, we conclude that $\frac{1}{2}c_4\cdot\varepsilon_0^{1/2}\cdot(d(A))^{91/400}\geqq c_5\cdot(\mu(A))^{89/800}\geqq$ $\geqq(\mu(A))^{1/9}$ if $\mu(A)$ is sufficiently large. Theorem 1.1 follows. ∎

3. Proof of Theorem 2.1—Part 1

Throughout we require $\mu(A)$ to be "sufficiently large" (i.e., bigger than a large absolute constant). Since $d(A)\geqq(\mu(A))^{1/2}$, it follows that $d(A)$ is also "sufficiently large". Without loss of generality we can assume that the inscribed circle of A is centered at the origin $\mathbf{0}\in\mathbf{R}^2$. Let χ_τ denote the characteristic function of the (rotated) region τA, i.e.,

$$\chi_\tau(\mathbf{x})=\begin{cases}1, & \text{if } \mathbf{x}\in\tau A\\ 0, & \text{if } \mathbf{x}\notin\tau A.\end{cases}$$

Let $N=N(d(A))\geqq 2$ be a large integer depending only on the diameter of A (N will be specified later). For notational convenience write $Q(N)=\left[-N-\frac{1}{2},\ N+\frac{1}{2}\right)^2$. Let μ_0 denote the restriction of the usual area μ to the square $Q(N)$, i.e., $\mu_0(S)=\mu(S\cap Q(N))=\text{area}\left(S\cap\left[-N-\frac{1}{2},\ N+\frac{1}{2}\right)^2\right)$, where

S is an arbitrary Lebesgue measurable set. We recall: $\mathcal{U}(\mathbf{n}, \varepsilon)=[-\varepsilon, \varepsilon)^2+\mathbf{n}$, $\mathbf{n}\in\mathbf{Z}^2$. For any $\varepsilon\in\left(0, \frac{1}{2}\right]$, let λ_ε denote the following measure:

$$\lambda_\varepsilon(S) = \frac{1}{4\varepsilon^2} \sum_{\mathbf{n}\in\mathbf{Z}^2\cap Q(N)} \mu(S\cap\mathcal{U}(\mathbf{n}, \varepsilon))$$

for all Lebesgue measurable sets $S\subset\mathbf{R}^2$. Note that $\lambda_{1/2}=\mu_0$. Let

$$F_{\tau,\varepsilon} = \chi_\tau*(\mathrm{d}\lambda_\varepsilon-\mathrm{d}\mu_0) \tag{3.1}$$

where $*$ denotes the *convolution* operation. More explicitly,

$$F_{\tau,\varepsilon}(\mathbf{x}) = \int_{\mathbf{R}^2} \chi_\tau(\mathbf{x}-\mathbf{y})\,(\mathrm{d}\lambda_\varepsilon(\mathbf{y})-\mathrm{d}\mu_0(\mathbf{y})) = \lambda_\varepsilon(\tau A+\mathbf{x})-\mu_0(\tau A+\mathbf{x}). \tag{3.2}$$

Observe that

$$F_{\tau,\varepsilon}(\mathbf{x}) = g(\tau A, \mathbf{x}, \varepsilon)-\mu(A) \quad \text{if} \quad \tau A+\mathbf{x}\subset Q(N) \quad \text{and} \tag{3.3}$$

$$F_{\tau,\varepsilon}(\mathbf{x}) = 0 \quad \text{if} \quad \tau A+\mathbf{x}\subset \mathbf{R}^2\setminus Q(N). \tag{3.4}$$

The basic idea of the proof is to utilize Fourier Analysis. We recall some facts from this theory. Given any $F\in L^2(\mathbf{R}^2)$, the expression

$$\hat{F}(\mathbf{t}) = \frac{1}{2\pi}\int_{\mathbf{R}^2} e^{-i\mathbf{x}\cdot\mathbf{t}}\cdot F(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

defines the *Fourier transform* of F (note that $i=\sqrt{-1}$ and $\mathbf{x}\cdot\mathbf{t}=x_1\cdot t_1+x_2\cdot t_2$ is the usual inner product). It is well known that $(F, G\in L^2(\mathbf{R}^2))$.

$$\widehat{F*G} = \hat{F}\cdot\hat{G} \quad (* \text{ is the convolution operation) and} \tag{3.5}$$

$$\int_{\mathbf{R}^2} |F(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} = \int_{\mathbf{R}^2} |\hat{F}(\mathbf{t})|^2\mathrm{d}\mathbf{t} \quad \text{(Plancherel identity).} \tag{3.6}$$

Now by (3.1), (3.5) and (3.6) we have the following key formula

$$\int_0^{2\pi}\int_{\mathbf{R}^2} (F_{\tau,\varepsilon}(\mathbf{x}))^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau = \int_0^{2\pi}\int_{\mathbf{R}^2} |\hat{F}_{\tau,\varepsilon}(\mathbf{t})|^2\,\mathrm{d}\mathbf{t}\,\mathrm{d}\tau = \tag{3.7}$$

$$= \int_{\mathbf{R}^2} \Big(\int_0^{2\pi} |\hat{\chi}_\tau(\mathbf{t})|^2\,\mathrm{d}\tau\Big)\cdot|(\mathrm{d}\hat{\lambda}_\varepsilon-\mathrm{d}\hat{\mu}_0)(\mathbf{t})|^2\,\mathrm{d}\mathbf{t}.$$

First we study the second factor in the right-hand side of (3.7). We have with $\mathbf{t}=(t_1, t_2)$,

$$(\mathrm{d}\hat{\lambda}_\varepsilon-\mathrm{d}\hat{\mu}_0)(\mathbf{t}) = \frac{1}{2\pi}\int_{\mathbf{R}^2} e^{-i\mathbf{x}\cdot\mathbf{t}}\mathrm{d}\lambda_\varepsilon(\mathbf{x})-\frac{1}{2\pi}\int_{\mathbf{R}^2} e^{-i\mathbf{x}\cdot\mathbf{t}}\,\mathrm{d}\mu_0(\mathbf{x}) =$$

$$= \frac{1}{2\pi}\Big(\sum_{n_1=-N}^{N}\frac{1}{2\varepsilon}\int_{n_1-\varepsilon}^{n_1+\varepsilon} e^{-ix_1\cdot t_1}\,\mathrm{d}x_1\Big)\Big(\sum_{n_2=N}^{N}\frac{1}{2\varepsilon}\int_{n_2-\varepsilon}^{n_2+\varepsilon} e^{-ix_2\cdot t_2}\,\mathrm{d}x_2\Big)-$$

$$-\frac{1}{2\pi}\Big(\sum_{n_1=-N}^{N}\int_{n_1-1/2}^{n_1+1/2} e^{-ix_1\cdot t_1}dx_1\Big)\Big(\sum_{n_2=-N}^{N}\int_{n_2-1/2}^{n_2+1/2} e^{-ix_2\cdot t_2}dx_2\Big)=$$

$$=\frac{1}{2\pi}\left(\sum_{n_1=-N}^{N}\frac{2\sin(\varepsilon\cdot t_1)}{2\varepsilon\cdot t_1}\cdot e^{-in_1\cdot t_1}\right)\left(\sum_{n_2=-N}^{N}\frac{2\sin(\varepsilon\cdot t_2)}{2\varepsilon\cdot t_2}\cdot e^{-in_2\cdot t_2}\right)-$$

$$-\left(\sum_{n_1=-N}^{N}\frac{2\sin\left(\frac{t_1}{2}\right)}{t_1}\cdot e^{-in_1\cdot t_1}\right)\left(\sum_{n_2=-N}^{N}\frac{2\sin\left(\frac{t_2}{2}\right)}{t_2}\cdot e^{-in_2\cdot t_2}\right).$$

Thus we have

$$(d\hat{\lambda}_\varepsilon-d\beta_0)(t)=$$

$$=\frac{1}{2\pi}\left(\frac{\sin(\varepsilon t_1)\cdot\sin(\varepsilon t_2)}{\varepsilon t_1\cdot\varepsilon t_2}-\frac{2\sin\left(\frac{t_1}{2}\right)\cdot 2\sin\left(\frac{t_2}{2}\right)}{t_1\cdot t_2}\right)\cdot\Big(\sum_{n_1=-N}^{N}e^{-in_1\cdot t_1}\Big)\Big(\sum_{n_2=-N}^{N}e^{-in_2\cdot t_2}\Big)=$$

$$=\frac{1}{2\pi}\left(\frac{\sin(\varepsilon t_1)\sin(\varepsilon t_2)}{\varepsilon t_1\varepsilon t_2}-\frac{\sin\left(\frac{t_1}{2}\right)\sin\left(\frac{t_2}{2}\right)}{\frac{t_1}{2}\cdot\frac{t_2}{2}}\right)\frac{\sin\left(\left(N+\frac{1}{2}\right)t_1\right)}{\sin\left(\frac{t_1}{2}\right)}\cdot\frac{\sin\left(\left(N+\frac{1}{2}\right)t_2\right)}{\sin\left(\frac{t_2}{2}\right)}.$$

We shall denote the distance from the real number ξ to the nearest integer by $\|\xi\|$. Suppose that $|t_1|\geqq\pi$, $|t_2|\geqq\pi$, $\left\|\frac{t_1}{2\pi}\right\|\leqq\frac{1}{4N+2}$ and $\left\|\frac{t_2}{2\pi}\right\|\leqq\frac{1}{4N+2}$. Let $\frac{t_1}{2\pi}=n\pm\left\|\frac{t_1}{2\pi}\right\|$, where n is an integer. Using the trivial inequality $1\geqq\frac{\sin x}{x}\geqq\frac{2}{\pi}$ for $|x|\leqq\frac{\pi}{2}$, we get

$$\frac{\sin\left(\left(N+\frac{1}{2}\right)t_1\right)}{\sin\left(\frac{t_1}{2}\right)}=\frac{\sin\left(\left(N+\frac{1}{2}\right)2\pi\left(n\pm\left\|\frac{t_1}{2\pi}\right\|\right)\right)}{\sin\left(\pi\left(n\pm\left\|\frac{t_1}{2\pi}\right\|\right)\right)}=$$

$$=\frac{\sin\left(\pi\cdot n\pm\pi(2N+1)\left\|\frac{t_1}{2\pi}\right\|\right)}{\sin\left(\pi\cdot n\pm\pi\left\|\frac{t_1}{2\pi}\right\|\right)}=\frac{\sin\left(\pi(2N+1)\left\|\frac{t_1}{2\pi}\right\|\right)}{\sin\left(\pi\left\|\frac{t_1}{2\pi}\right\|\right)}\geqq$$

$$\geqq\frac{2}{\pi}\frac{\pi(2N+1)\left\|\frac{t_1}{2\pi}\right\|}{\sin\left(\pi\left\|\frac{t_1}{2\pi}\right\|\right)}\geqq\frac{2}{\pi}\frac{\pi(2N+1)\left\|\frac{t_1}{2\pi}\right\|}{\left\|\frac{t_1}{2\pi}\right\|}=\frac{2}{\pi}(2N+1)>N.$$

Therefore,

$$\frac{\sin\left[\left(N+\frac{1}{2}\right)t_1\right]\cdot\sin\left[\left(N+\frac{1}{2}\right)t_2\right]}{\sin\left(\frac{t_1}{2}\right)\cdot\sin\left(\frac{t_2}{2}\right)} > N^2.$$

Suppose further that $|\varepsilon\cdot t_1|\leqq\frac{\pi}{2}$, $|\varepsilon\cdot t_2|\leqq\frac{\pi}{2}$ and $N\geqq 2$. Then

$$\frac{\sin(\varepsilon\cdot t_1)}{\varepsilon\cdot t_1}\cdot\frac{\sin(\varepsilon\cdot t_2)}{\varepsilon\cdot t_2}-\frac{\sin\left(\frac{t_1}{2}\right)}{\frac{t_1}{2}}\cdot\frac{\sin\left(\frac{t_2}{2}\right)}{\frac{t_2}{2}}\geqq$$

$$\geqq\left(\frac{\sin(\varepsilon t_1)}{\varepsilon t_1}-\frac{\sin\left(\frac{t_1}{2}\right)}{\frac{t_1}{2}}\right)\cdot\frac{\sin(\varepsilon t_2)}{\varepsilon t_2}\geqq$$

$$\geqq\left(\frac{2}{\pi}-\frac{\sin\left(\pi\left\|\frac{t_1}{2\pi}\right\|\right)}{\frac{t_1}{2}}\right)\cdot\frac{2}{\pi}\geqq\left(\frac{2}{\pi}-\frac{\sin\left(\frac{\pi}{4N+2}\right)}{\frac{\pi}{2}}\right)\cdot\frac{2}{\pi}\geqq$$

$$\geqq\left(\frac{2}{\pi}-\frac{\frac{\pi}{4N+2}}{\frac{\pi}{2}}\right)\cdot\frac{2}{\pi}=\left(\frac{2}{\pi}-\frac{1}{2N+1}\right)\cdot\frac{2}{\pi}>\frac{1}{10}.$$

Summarizing, we conclude that

$$|(\mathrm{d}\hat{\lambda}_\varepsilon-\mathrm{d}\hat{\mu}_0)(\mathbf{t})| = c_6\cdot N^2 \quad \text{whenever} \tag{3.8}$$

$$|t_1-2\pi k|+|t_2-2\pi l| \leqq \frac{\pi}{2N+1} \quad \text{for some integers} \quad k,\, l\in\left[1,\frac{1}{5\varepsilon}\right].$$

Next we investigate the first factor $\int\limits_0^{2\pi}|\hat{\chi}_\tau(\mathbf{t})|^2\,\mathrm{d}\tau$ in the right-hand side of (3.7). Using the trivial identities $\hat{\chi}_\tau(\mathbf{t})=\hat{\chi}_A(\tau^{-1}\mathbf{t})$ and $\hat{\chi}_{A+\mathbf{v}}(\mathbf{t}')=e^{-i\mathbf{t}'\cdot\mathbf{v}}\cdot\hat{\chi}_A(\mathbf{t}')$, $\mathbf{v}\in\mathbf{R}^2$, where χ_A and $\chi_{A+\mathbf{v}}$ denote the characteristic function of the given convex region A and its translate $A+\mathbf{v}$, resp., we obtain that

$$\int\limits_0^{2\pi}|\hat{\chi}_\tau(\mathbf{t})|^2\mathrm{d}\tau = \int\limits_0^{2\pi}\left(\frac{1}{\pi}\int\limits_{|\mathbf{v}|\leqq 1}|\hat{\chi}_{A+\mathbf{v}}(\tau^{-1}\mathbf{t})|^2\,\mathrm{d}\mathbf{v}\right)\mathrm{d}\tau \tag{3.9}$$

where $\{\mathbf{v}: |\mathbf{v}|\leqq 1\}=\{\mathbf{v}=(v_1, v_2): v_1^2+v_2^2\leqq 1\}$ is the unit disc. For any $\beta\in[0, 2\pi)$, write $\mathbf{e}(\beta)=(\cos\beta, \sin\beta)$. Let $h_{A+\mathbf{v}}(\beta, y)$ be the Euclidean length of the chord $\{\mathbf{y}\in A+\mathbf{v}: \mathbf{y}\cdot\mathbf{e}(\beta)=y\}$. We say that $h_{A+\mathbf{v}}(\beta, y)$, $\beta\in[0, 2\pi)$, $y\in\mathbf{R}$ is the *chord function* of $A+\mathbf{v}$. Let $\mathbf{s}=s\cdot\mathbf{e}(\beta)$, $s\in\mathbf{R}$. We then have

$$(3.10)\qquad \hat{\chi}_{A+\mathbf{v}}(\mathbf{s}) = \frac{1}{2\pi}\int_{A+\mathbf{v}} e^{-i\mathbf{x}\cdot\mathbf{s}}\,d\mathbf{x} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixs}\cdot h_{A+\mathbf{v}}(\beta, x)\,dx = \frac{1}{2\pi}H_{\beta,\mathbf{v}}(s)$$

where $H_{\beta,\mathbf{v}}(s)=\int_{-\infty}^{\infty} e^{-ixs}\cdot h_{A+\mathbf{v}}(\beta, x)\,dx$. Returning to (3.9), by (3.10) we have

(3.11)

$$\int_0^{2\pi} |\hat{\chi}_\tau(\mathbf{t})|^2\,d\tau = \int_0^{2\pi}\left(\frac{1}{\pi}\int_{|\mathbf{v}|\leqq 1} |\hat{\chi}_{A+\mathbf{v}}(\tau^{-1}\mathbf{t})|^2\,d\mathbf{v}\right)d\tau = \int_0^{2\pi}\frac{1}{(2\pi)^2}\int_{|\mathbf{v}|\leqq 1}(H_{\beta,\mathbf{v}}(|\mathbf{t}|))^2\,d\mathbf{v}\,d\beta.$$

Now let $E=\left[\frac{1}{5\varepsilon}\right]$ (integral part), and for all integers $k, l\in[1, E]$, write $T(k, l)=\left\{\mathbf{t}=(t_1, t_2): |t_1-2\pi k|+|t_2-2\pi l|\leqq\frac{\pi}{2N+1}\right\}$ and $P(k, l)=2\pi\cdot(k^2+l^2)^{1/2}$. By (3.7), (3.8) and (3.11) we obtain that

$$(3.12)\qquad \int_0^{2\pi}\int_{\mathbf{R}^2} F_{\tau,\varepsilon}^2(\mathbf{x})\,d\mathbf{x}\,d\tau \geqq$$

$$\geqq \sum_{k=1}^{E}\sum_{l=1}^{E}\int_{T(k,l)} |(d\hat{\lambda}_\varepsilon - d\hat{\mu}_0)(\mathbf{t})|^2\cdot\left(\int_0^{2\pi}|\hat{\chi}_\tau(\mathbf{t})|^2\,d\tau\right)d\mathbf{t} \geqq$$

$$\geqq (c_6\cdot N^2)^2\cdot\sum_{k=1}^{E}\sum_{l=1}^{E}\int_{T(k,l)}\int_0^{2\pi}|\hat{\chi}_\tau(\mathbf{t})|^2\,d\tau\,d\mathbf{t} \geqq$$

$$\geqq c_7\cdot N^4\cdot\sum_{k=1}^{E}\sum_{l=1}^{E}\int_{T(k,l)}\int_0^{2\pi}\int_{|\mathbf{v}|\leqq 1}(H_{\beta,\mathbf{v}}(|\mathbf{t}|))^2\,d\mathbf{v}\,d\beta\,d\mathbf{t} \geqq$$

$$\geqq c_8\cdot N^3\cdot\sum_{k=1}^{E}\sum_{l=1}^{E}\int_{P(k,l)-1/N}^{P(k,l)+1/N}\int_0^{2\pi}\int_{|\mathbf{v}|\leqq 1} H_{\beta,\mathbf{v}}^2(s)\,d\mathbf{v}\,d\beta\,ds.$$

Let $M_{\beta,\mathbf{v}}^+=\sup\{x\in\mathbf{R}: h_{A+\mathbf{v}}(\beta, x)>0\}$ and $M_{\beta,\mathbf{v}}^-=\inf\{x\in\mathbf{R}: h_{A+\mathbf{v}}(\beta, x)>0\}$. Clearly $D_\beta=(M_{\beta,\mathbf{v}}^+-M_{\beta,\mathbf{v}}^-)$ is the length of the projection of A onto a straight line parallel to the unit vector $\mathbf{e}(\beta)$. Since

$$H_{\beta,\mathbf{v}}(s) = \int_{M_{\beta,\mathbf{v}}^-}^{M_{\beta,\mathbf{v}}^+} e^{-ixs}\cdot h_{A+\mathbf{v}}(\beta, x)\,dx = \int_{M_{\beta,\mathbf{v}}^-}^{M_{\beta,\mathbf{v}}^+}(\cos(xs)-i\cdot\sin(xs))\cdot h_{A+\mathbf{v}}(\beta, x)\,dx,$$

we have

$$(3.13)\qquad |H_{\beta,\mathbf{v}}(s)| \geqq \left| \int_{M_{\beta,\mathbf{v}}^-}^{M_{\beta,\mathbf{v}}^+} \cos(xs)\cdot h_{A+\mathbf{v}}(\beta, x)\,\mathrm{d}x \right|.$$

It suffices to prove Theorem 2.1 for convex regions with analytic boundary arc. Indeed, given any convex region A one can find an inscribed convex region $A'\subset A$ such that $\Gamma(A')$ is an analytic curve and $\mu(A\setminus A')\leqq 1$. Thus we can assume that $\Gamma(A)$ is sufficiently smooth. We have

$$\int_{M_{\beta,\mathbf{v}}^-}^{M_{\beta,\mathbf{v}}^+} \cos(xs)\cdot h_{A+\mathbf{v}}(\beta, x)\,\mathrm{d}x = -\int_{M_{\beta,\mathbf{v}}^-}^{M_{\beta,\mathbf{v}}^+} \frac{\sin(xs)}{s}\cdot\frac{\partial h_{A+\mathbf{v}}(\beta, x)}{\partial x}\,\mathrm{d}x,$$

and so by (3.13),

$$(3.14)\qquad |H_{\beta,\mathbf{v}}(s)| \geqq \left| \int_{M_{\beta,\mathbf{v}}^-}^{M_{\beta,\mathbf{v}}^+} \frac{\sin(xs)}{s}\cdot\frac{\partial h_{A+\mathbf{v}}(\beta, x)}{\partial x}\,\mathrm{d}x \right|.$$

For later purpose we note that the partial derivative

$$\frac{\partial h_{A+\mathbf{v}}(\beta, x)}{\partial x}$$

is monotonically decreasing in the interval $M_{\beta,\mathbf{v}}^- < x < M_{\beta,\mathbf{v}}^+$.

Let $\varepsilon=\varepsilon_0=(d(A))^{-(1/100)}$. Let $\eta\in\left(0, \frac{1}{100}\right)$. The parameter η will be specified later as a sufficiently small positive absolute constant. Let $\{\xi\}$ denote the *fractional part* of the real number ξ, i.e., $\xi=[\xi]+\{\xi\}$.

For any $\beta\in[0, 2\pi)$, write $V(\beta)=V_\eta(\beta)=\{\mathbf{v}\in\mathbf{R}^2\colon |\mathbf{v}|\leqq 1$ and one can find positive integers $k=k(\beta, \mathbf{v})$, $l=l(\beta, \mathbf{v})$ such that

$$(3.15)\qquad \frac{1}{10\varepsilon_0} \leqq (k^2+l^2)^{1/2} \leqq \frac{1}{5\varepsilon_0}, \quad \text{and furthermore,}$$

$$(3.16)\qquad \|(k^2+l^2)^{1/2}\cdot M_{\beta,\mathbf{v}}^-\| \leqq 3\eta \quad\text{and}\quad \eta \leqq \{(k^2+l^2)^{1/2} M_{\beta,\mathbf{v}}^+\} \leqq 2\eta\}.$$

For any $\beta\in[0, 2\pi)$ and $\mathbf{v}\in V(\beta)$, let $k_0(\beta, \mathbf{v})$ and $l_0(\beta, \mathbf{v})$ denote integers k and l satisfying relations (3.15) and (3.16). Moreover, let $m_0^-(\beta, \mathbf{v})$ and $m_0^+(\beta, \mathbf{v})$ be integers satisfying

$$(3.17)\qquad \left|(k_0^2(\beta, \mathbf{v})+l_0^2(\beta, \mathbf{v}))^{1/2}\cdot M_{\beta,\mathbf{v}}^- - m_0^-(\beta, \mathbf{v})\right| \leqq 3\cdot\eta \quad\text{and}$$

$$(3.18)\qquad \eta \leqq (k_0^2(\beta, \mathbf{v})+l_0^2(\beta, \mathbf{v}))^{1/2}\cdot M_{\beta,\mathbf{v}}^+ - m_0^+(\beta, \mathbf{v}) \leqq 2\eta.$$

Let $P(k, l)=2\pi\cdot(k^2+l^2)^{1/2}$ and $P(k_0, l_0)=2\pi\big(k_0^2(\beta, \mathbf{v})+l_0^2(\beta, \mathbf{v})\big)^{1/2}$. By (3.12) and (3.14) we have

$$\int_0^{2\pi}\int_{\mathbf{R}^2} F^2_{\tau,\varepsilon_0}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau \geqq \tag{3.19}$$

$$\geqq c_8\cdot N^3\cdot\sum_{k=1}^{E}\sum_{l=1}^{E}\int_{P(k,l)-1/N}^{P(k,l)+1/N}\int_0^{2\pi}\int_{|\mathbf{v}|\leqq 1} H^2_{\beta,\mathbf{v}}(s)\,\mathrm{d}\mathbf{v}\,\mathrm{d}\beta\,\mathrm{d}s \geqq$$

$$\geqq c_8\cdot N^3\cdot\int_0^{2\pi}\int_{V(\beta)}\int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N}\left(\int_{M^-_{\beta,\mathbf{v}}}^{M^+_{\beta,\mathbf{v}}}\frac{\sin(xs)}{s}\cdot\frac{\partial h_{A+\mathbf{v}}(\beta, x)}{\partial x}\,\mathrm{d}x\right)^2\mathrm{d}s\,\mathrm{d}\mathbf{v}\,\mathrm{d}\beta.$$

In the rest of the proof we shall give a lower bound to the right-hand side of (3.19).

Let $\beta\in[0, 2\pi)$ and $\mathbf{v}\in\mathbf{R}^2$, $|\mathbf{v}|\leqq 1$ be arbitrary but fixed. For notational convenience, write $M^-=M^-_{\beta,\mathbf{v}}$, $M^+=M^+_{\beta,\mathbf{v}}$, $k_0=k_0(\beta, \mathbf{v})$, $l_0=l_0(\beta, \mathbf{v})$, $P(k_0, l_0)=2\pi\big(k_0^2(\beta, \mathbf{v})+l_0^2(\beta, \mathbf{v})\big)^{1/2}$, $m_0^-=m_0^-(\beta, \mathbf{v})$, $m_0^+=m_0^+(\beta, \mathbf{v})$ and $h(x)=h_{A+\mathbf{v}}(\beta, x)$. Let $s\in\left[P(k_0, l_0)-\frac{1}{N},\ P(k_0, l_0)+\frac{1}{N}\right]$. Then by (3.17) we have

$$|s\cdot M^- - 2\pi m_0^-| \leqq \tag{3.20}$$

$$\leqq |s\cdot M^- - P(k_0, l_0)\cdot M^-| + |P(k_0, l_0)M^- - 2\pi m_0^-| \leqq \frac{|M^-|}{N}+2\pi\cdot 3\eta.$$

Since the inscribed circle of A is centered at the origin $0\in\mathbf{R}^2$, we have $|M^-|=|M^-_{\beta,\mathbf{v}}|=|M^-_{\beta,0}+\mathbf{e}(\beta)\cdot\mathbf{v}|\leqq d(A)+1$. Thus by (3.20),

$$|s\cdot M^- - 2\pi m_0^-| \leqq 6\pi\eta+\frac{d(A)+1}{N} < 20\eta \quad \text{provided} \tag{3.21}$$

$$N \geqq \frac{d(A)+1}{\eta}. \tag{3.22}$$

Similarly, by (3.18),

$$0 < 2\pi\eta-\eta \leqq s\cdot M^+ - 2\pi m_0^+ \leqq 4\pi\eta+\eta < 14\eta \quad \text{provided (3.22) holds.} \tag{3.23}$$

Let $0<\gamma<\delta$ be real numbers with

$$\frac{40\eta}{s} < \gamma < \delta < \frac{\frac{\pi}{2}-20\eta}{s} \tag{3.24}$$

(the exact values of γ and δ will be specified later). We define a partition $[M^-, M^+]=$

$=\bigcup_{j=1}^{7} I_j$ as follows. Let

$$I_1 = I_1(\beta, \mathbf{v}) = [M^-, M^- + \gamma)$$

$$I_2 = I_2(\beta, \mathbf{v}) = [M^- + \gamma, M^- + \delta)$$

$$I_3 = I_3(\beta, \mathbf{v}, s) = \left[M^- + \delta,\, 2\pi \cdot \frac{\left(m_0^- + \frac{1}{2}\right)}{s}\right)$$

$$I_4 = I_4(\beta, \mathbf{v}, s) = \left[2\pi \cdot \frac{\left(m_0^- + \frac{1}{2}\right)}{s},\, M^- + \frac{\pi}{s} + \delta\right)$$

$$I_5 = I_5(\beta, \mathbf{v}, s) = \left[M^- + \frac{\pi}{s} + \delta,\, 2\pi \cdot \frac{(m_0^- + 1)}{s}\right)$$

$$I_6 = I_6(\beta, \mathbf{v}, s) = \left[2\pi \cdot \frac{(m_0^- + 1)}{s},\, 2\pi \cdot \frac{m_0^+}{s}\right]$$

$$I_7 = I_7(\beta, \mathbf{v}, s) = \left(2\pi \cdot \frac{m_0^+}{s},\, M^+\right].$$

From (3.21), (3.23) and (3.24) it follows that

$$(3.25) \qquad 2\pi \cdot \frac{m_0^-}{s} + \frac{\gamma}{2} < M^- + \gamma < 2\pi \cdot \frac{m_0^-}{s} + \frac{3}{2}\gamma, \quad M^- + \delta < 2\pi \cdot \frac{m_0^-}{s} + \frac{\pi}{2s} \quad \text{and}$$

$$(3.26) \qquad \text{length}\,(I_7) < \frac{14\eta}{s}.$$

Note that I_6 is well defined, since by (3.15), (3.17) and (3.18), $m_0^+ - m_0^- \geqq (k_0^2 + l_0^2)^{1/2} \times$ $\times (M^+ - M^-) - 5\eta \geqq \frac{1}{10\varepsilon_0} \cdot 2r(A) - 5\eta \geqq (d(A))^{1/100} \cdot \frac{2}{90} - 1 \geqq 2$ if $\mu(A)$ is sufficiently large. By (3.19) we have

$$(3.27) \qquad \int_0^{2\pi} \int_{\mathbf{R}^2} F_{\tau,\varepsilon_0}^2(\mathbf{x})\, d\mathbf{x}\, d\tau \geqq$$

$$\geqq c_8 \cdot N^3 \cdot \int_0^{2\pi} \int_{V(\beta)} \int_{P(k_0, l_0) - 1/N}^{P(k_0, l_0) + 1/N} \left(\sum_{j=1}^{7} \int_{I_j} \frac{\sin(xs)}{s} \cdot \frac{\partial h(x)}{\partial x}\, dx\right)^2 ds\, d\mathbf{v}\, d\beta.$$

Since the partial derivative

$$\frac{\partial h(x)}{\partial x} = \frac{\partial h_{A+\mathbf{v}}(\beta, x)}{\partial x}$$

is monotonically decreasing in the interval $M^- = M^-_{\beta,\mathbf{v}} < x < M^+ = M^+_{\beta,\mathbf{v}}$, we have

$$\int_{I_6} \frac{\sin(xs)}{s} \cdot \frac{\partial h(x)}{\partial x} \, dx = \sum_{n=m_0^-+1}^{m_0^+-1} \int_{2\pi(n/s)}^{2\pi((n+1)/s)} \frac{\sin(xs)}{s} \cdot \frac{\partial h(x)}{\partial x} \, dx = \tag{3.28}$$

$$= \sum_{n=m_0^-+1}^{m_0^+-1} \int_{2\pi(n/s)}^{2\pi((n+1/2)/s)} \frac{\sin(xs)}{s} \cdot \left(\frac{\partial h(x)}{\partial x} - \frac{\partial h\left(x+\frac{\pi}{s}\right)}{\partial x} \right) dx \geqq 0.$$

Similarly,

$$\sum_{j=3,5} \int_{I_j} \frac{\sin(xs)}{s} \cdot \frac{\partial h(x)}{\partial x} \, dx = \int_{I_3} \frac{\sin(xs)}{s} \cdot \left(\frac{\partial h(x)}{\partial x} - \frac{\partial h\left(x+\frac{\pi}{s}\right)}{\partial x} \right) dx \geqq 0. \tag{3.29}$$

Let

$$|a|^+ = \begin{cases} a, & \text{if } a > 0 \\ 0, & \text{if } a \leqq 0 \end{cases} \quad \text{and} \quad |a|^- = \begin{cases} a, & \text{if } a < 0 \\ 0, & \text{if } a \geqq 0. \end{cases}$$

Then by (3.27)—(3.29) we have

$$\int_0^{2\pi} \int_{\mathbf{R}^2} F^2_{\tau,\varepsilon_0}(\mathbf{x}) \, d\mathbf{x} \, d\tau \geqq \tag{3.30}$$

$$\geqq c_8 \cdot N^3 \cdot \int_0^{2\pi} \int_{V(\beta)} \int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N} \left(\left| \sum_{j=1,2,4,7} \int_{I_j} \frac{\sin(xs)}{s} \cdot \frac{\partial h(x)}{\partial x} \, dx \right|^+ \right)^2 ds \, d\mathbf{v} \, d\beta.$$

4. Proof of Theorem 2.1—Part 2

The second part of the proof is based on the following two lemmas (we use the same notation as in Section 3).

Lemma 4.1. *If* $\frac{1}{100} \geqq \eta \geqq 2 \cdot (d(A))^{-10^{-5}}$ *and* $\mu(A)$ *is larger than an "ineffective" absolute constant, then* $\mu(V(\beta)) = \mu(V_\eta(\beta)) \geqq \eta$ *uniformly for all* $\beta \in [0, 2\pi)$.

The second one is a purely geometric lemma.

Given a convex region B, an angle $\beta \in [0, 2\pi)$ and a real number $y \geqq 0$, write

$$\begin{aligned} f_B(\beta, y) &= h_{B+\mathbf{v}}(\beta, M^-_{\beta,\mathbf{v}} + y) \quad \text{where} \\ M^-_{\beta,\mathbf{v}} &= M^-_{\beta,\mathbf{v}}(B) = \inf\{x \in \mathbf{R}: h_{B+\mathbf{v}}(\beta, x) > 0\}. \end{aligned} \tag{4.1}$$

Observe that the right-hand side term in (4.1) is independent of the value of $\mathbf{v} \in \mathbf{R}^2$.

Lemma 4.2. *There are ("effective") positive absolute constants* c_9, c_{10} *and* c_{11} *such that for any convex region* B *with* $r(B) \geqq c_9$,

$$c_{10} \cdot d(B) \geqq \int_0^{2\pi} (f_B(\beta, 1))^2 \, d\beta \geqq c_{11} \cdot d(B).$$

We postpone the proofs to Sections 5—7.
Next we state three corollaries of Lemma 4.2.

Lemma 4.2A. *Let* A *be a convex region with* $r(A) \geqq \frac{1}{9}$. *If* $0 < y \leqq \frac{1}{9 \cdot c_9}$ *the*

$$c_{10} \cdot y \cdot d(A) \geqq \int_0^{2\pi} (f_A(\beta, y))^2 \, d\beta \geqq c_{11} \cdot y \cdot d(A).$$

Proof. Let $B = \frac{1}{y} A = \left\{ \frac{1}{y} \mathrm{x} \colon \ \mathrm{x} \in A \right\}$. Then $r(B) = \frac{1}{y} \cdot r(A) \geqq 9 \cdot c_9 \cdot \frac{1}{9} = c_9$. Thus by Lemma 4.2 we have $c_{10} \cdot d(B) \geqq \int_0^{2\pi} (f_B(\beta, 1))^2 \, d\beta \geqq c_{11} \cdot d(B)$. Since $d(B) = \frac{1}{y} \cdot d(A)$ and $f_B(\beta, 1) = \frac{1}{y} f_A(\beta, y)$, Lemma 4.2A follows. ∎

Let $\Omega(y) = \Omega(A, y) = \{\beta \in [0, 2\pi) \colon \ f_A(\beta, y) = \max_{0 \leqq x \leqq y} f_A(\beta, x)\}$.

Lemma 4.2B. *Let* A *be a convex region with* $(rA) \geqq \frac{1}{9}$. *If* $0 < y \leqq \frac{c_{11}}{18 \cdot c_9 \cdot c_{10}}$, *then*

$$\int_{\Omega(y)} (f_A(\beta, y))^2 \, d\beta \geqq \frac{1}{4} \cdot \frac{(c_{11})^2}{c_{10}} \cdot y \cdot d(A).$$

Proof. Let $z = 2 \cdot \frac{c_{10}}{c_{11}} \cdot y$. We have $z \leqq 2 \cdot \frac{c_{10}}{c_{11}} \cdot \frac{c_{11}}{18 \cdot c_9 \cdot c_{10}} = \frac{1}{9 \cdot c_9}$. Thus by Lemma 4.2A,

$$\int_0^{2\pi} (f_A(\beta, z))^2 \, d\beta \geqq c_{11} \cdot z \cdot d(A) = 2c_{10} \cdot y \cdot d(A) \geqq 2 \int_0^{2\pi} (f_A(\beta, y))^2 \, d\beta. \tag{4.2}$$

We need the following two consequences of the convexity of A:

$$f_A(\beta, z) \leqq f_A(\beta, y) \quad \text{for all} \quad \beta \in \bar{\Omega}(y) = [0, 2\pi) \setminus \Omega(y), \tag{4.3}$$

$$f_A(\beta, y) \geqq \frac{y}{z} f_A(\beta, z) \quad \text{for all} \quad \beta \in [0, 2\pi). \tag{4.4}$$

Combining (4.2) and (4.3), we obtain that

$$\begin{aligned} \int_{\Omega(y)} (f_A(\beta, z))^2 \, d\beta &= \int_0^{2\pi} (f_A(\beta, z))^2 \, d\beta - \int_{\bar{\Omega}(y)} (f_A(\beta, z))^2 \, d\beta \geqq \\ &\geqq \int_0^{2\pi} (f_A(\beta, z))^2 \, d\beta - \int_{\bar{\Omega}(y)} (f_A(\beta, y))^2 \, d\beta \geqq \\ &\geqq \int_0^{2\pi} (f_A(\beta, z))^2 \, d\beta - \int_0^{2\pi} (f_A(\beta, y))^2 \, d\beta \geqq \frac{1}{2} \int_0^{2\pi} (f_A(\beta, z))^2 \, d\beta. \end{aligned} \tag{4.5}$$

Now from (4.4), (4.5) and Lemma 4.2A it follows that

$$\int_{\Omega(y)} (f_A(\beta, y))^2 \, d\beta \geqq \int_{\Omega(y)} \left(\frac{y}{z} f_A(\beta, z)\right)^2 d\beta =$$

$$= \left(\frac{y}{z}\right)^2 \cdot \int_{(\Omega)y} (f_A(\beta, z))^2 \, d\beta \geqq \frac{1}{2}\left(\frac{y}{z}\right)^2 \cdot \int_0^{2\pi} (f_A(\beta, z))^2 \, d\beta \geqq$$

$$\geqq \frac{1}{2} \cdot \left(\frac{y}{z}\right)^2 \cdot c_{11} \cdot z \cdot d(A) = \frac{1}{4} \cdot \frac{(c_{11})^2}{c_{10}} \cdot y \cdot d(A).$$

This completes the proof of Lemma 4.2B. ▮

Lemma 4.2C. *Let A be a convex region with $r(A) \geqq \frac{1}{9}$. If $0 < y \leqq \min\left\{\frac{1}{9}, \frac{1}{9 \cdot c_9}\right\}$, then*

$$\int_0^{2\pi} (\max_{0 \leqq x \leqq y} f_A(\beta, x))^2 \, d\beta \leqq 4c_{10} \cdot y \cdot d(A).$$

Proof. We recall: $D_\beta = M^+_{\beta,v} - M^-_{\beta,v}$. From the convexity of A it follows that for any real numbers $0 \leqq x \leqq y \leqq \frac{1}{9} \leqq r(A)$,

$$\frac{f_A(\beta, y)}{f_A(\beta, x)} \geqq \frac{D_B - y}{D_\beta - x} \geqq \frac{D_\beta - r(A)}{D_\beta} \geqq \frac{\frac{1}{2} D_\beta}{D_\beta} = \frac{1}{2}.$$

Thus, by Lemma 4.2A we conclude that

$$\int_0^{2\pi} (\max_{0 \leqq x \leqq y} f_A(\beta, x))^2 \, d\beta \leqq \int_0^{2\pi} (2 f_A(\beta, y))^2 \, d\beta \leqq 4 \cdot c_{10} \cdot y \cdot d(A),$$

and Lemma 4.2C follows. ▮

We return to Lemma 4.1. Since η will be fixed as a small positive absolute constant, by Lemma 4.1 we have for all $\beta \in [0, 2\pi)$,

(4.6) $$\pi \geqq \mu(V(\beta)) \geqq \eta$$

provided $\mu(A)$ is sufficiently large.

We recall: the real parameter δ satisfies inequality (3.24) (the exact value of δ will be specified later). For $j = 1, 2, 3, \ldots$, write $\Omega_j(\delta) = \Omega_j(A, \delta) = \{\beta \in \Omega(A, \delta) \colon 2^j \cdot \eta > \mu(V(\beta)) \geqq 2^{j-1} \cdot \eta\}$. By (4.6), we have

(4.7) $$\Omega(\delta) = \bigcup_{1 \leqq j \leqq c_{12} \cdot \log(1/\eta)} \Omega_j(\delta).$$

We recall (3.15) and (3.24): $0<\gamma<\delta<\frac{\pi}{2s}$ where $s\in\left[P(k_0, l_0)-\frac{1}{N}, P(k_0, l_0)+\frac{1}{N}\right]$, $P(k_0, l_0)=2\pi\cdot(k_0^2(\beta, \mathbf{v})+l_0^2(\beta, \mathbf{v}))^{1/2}$, $\frac{1}{10\varepsilon_0}\leqq(k_0^2(\beta, \mathbf{v})+l_0^2(\beta, \mathbf{v}))^{1/2}\leqq\frac{1}{5\varepsilon_0}$ and $\varepsilon_0=(d(A))^{-(1/100)}$. Hence

$$0<\gamma<\delta<\frac{\pi}{2s}\leqq\min\left\{\frac{1}{9}, \frac{1}{9\cdot c_9}, \frac{c_{11}}{18\cdot c_9\cdot c_{10}}\right\} \tag{4.8}$$

provided $\mu(A)$ is sufficiently large. Thus by Lemma 4.2B,

$$\int_{\Omega(\delta)}(f_A(\beta, \delta))^2\,d\beta\geqq\frac{1}{4}\cdot\frac{(c_{11})^2}{c_{10}}\cdot\delta\cdot d(A). \tag{4.9}$$

From (4.7) and (4.9) it follows that for some $\nu\in\left[1, c_{12}\cdot\log\left(\frac{1}{\eta}\right)\right]$,

$$\int_{\Omega_\nu(\delta)}(f_A(\beta, \delta))^2\,d\beta\geqq\frac{(c_{11})^2}{4\cdot c_{12}\cdot\log\left(\frac{1}{\eta}\right)\cdot c_{10}}\cdot\delta\cdot d(A). \tag{4.10}$$

Now we return to (3.30).
Let

$$Z_2^+=\int_{\Omega_\nu(\delta)}\int_{V(\beta)}\int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N}\left(\left|\int_{I_2}\frac{\sin(x\cdot s)}{s}\cdot\frac{\partial h(x)}{\partial x}\,dx\right|^+\right)^2 ds\,d\mathbf{v}\,d\beta, \tag{4.11}$$

$$Z_4^-=\int_{\Omega_\nu(\delta)}\int_{V(\beta)}\int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N}\left(\left|\int_{I_4}\frac{\sin(x\cdot s)}{s}\cdot\frac{\partial h(x)}{\partial x}\,dx\right|^-\right)^2 ds\,d\mathbf{v}\,d\beta \tag{4.12}$$

and for $j=1, 7$ let

$$Z_j=\int_{\Omega_\nu(\delta)}\int_{V(\beta)}\int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N}\left(\int_{I_j}\frac{\sin(x\cdot s)}{s}\cdot\frac{\partial h(x)}{\partial x}\,dx\right)^2 ds\,d\mathbf{v}\,d\beta. \tag{4.13}$$

Using the elementary inequality (x, y real numbers) $(|x+y|^+)^2\geqq\frac{1}{2}(|x|^+)^2-(|y|^-)^2$, we have ($a_1, a_2, a_4, a_7$ real numbers)

$$\begin{aligned}(|a_1+a_2+a_4+a_7|^+)^2 &\geqq \frac{1}{2}(|a_2+a_4|^+)^2-(|a_1+a_7|^-)^2\geqq\\ &\geqq\frac{1}{2}\left(\frac{1}{2}(|a_2|^+)^2-(|a_4|^-)^2\right)-(|a_1|+|a_7|)^2\geqq\\ &\geqq\frac{1}{4}(|a_2|^+)^2-\frac{1}{2}(|a_4|^-)^2-2(a_1)^2-2(a_7)^2.\end{aligned} \tag{4.14}$$

Combining (3.30), (4.11)—(4.14) it follows that

$$\int_0^{2\pi}\int_{\mathbf{R}^2}(F_{\tau,\varepsilon_0}(\mathbf{x}))^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau \geqq \tag{4.15}$$

$$\geqq c_8\cdot N^3\int_{\Omega_\nu(\delta)}\int_{V(\beta)}\int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N}\left(\left|\sum_{j=1,2,4,7}\int_{I_j}\frac{\sin(x\cdot s)}{s}\cdot\frac{\partial h(x)}{\partial x}\mathrm{d}x\right|^+\right)^2\mathrm{d}s\,\mathrm{d}\mathbf{v}\,\mathrm{d}\beta \geqq$$

$$\geqq c_8\cdot N^3\left(\frac{1}{4}Z_2^+-\frac{1}{2}Z_4^- -2Z_1-2Z_7\right).$$

We are going to estimate the terms Z_2^+, Z_4^-, Z_1 and Z_7. First we give a lower bound to Z_2^+. By (4.1) we have for any $\beta\in\Omega(\delta)$, $\frac{\partial f_A(\beta, x-M^-)}{\partial x}=\frac{\partial h(x)}{\partial x}\geqq 0$ whenever $M^-\leqq x\leqq M^-+\delta$, therefore, by (3.25),

$$\left|\int_{I_2}\frac{\sin(x\cdot s)}{s}\cdot\frac{\partial h(x)}{\partial x}\mathrm{d}x\right|^+=\int_{M^-+\gamma}^{M^-+\delta}\frac{\sin(x\cdot s)}{s}\cdot\frac{\partial h(x)}{\partial x}\mathrm{d}x\geqq \tag{4.16}$$

$$\geqq\left(\min_{M^-+\gamma\leqq x\leqq M^-+\delta}\sin(x\cdot s)\right)\cdot\frac{1}{s}\int_{M^-+\gamma}^{M^-+\delta}\frac{\partial h(x)}{\partial x}\mathrm{d}x\geqq\frac{\sin\left(\gamma\cdot\frac{s}{2}\right)}{s}\int_{M^-+\gamma}^{M^-+\delta}\frac{\partial h(x)}{\partial x}\mathrm{d}x=$$

$$=\frac{\sin\left(\gamma\cdot\frac{s}{2}\right)}{s}\left(h(M^-+\delta)-h(M^-+\gamma)\right)=\frac{\sin\left(\gamma\cdot\frac{s}{2}\right)}{s}\left(f_A(\beta,\delta)-f_A(\beta,\gamma)\right)\geqq 0.$$

Thus, by (4.11) and (4.16),

$$Z_2^+\geqq\int_{\Omega_\nu(\delta)}\int_{V(\beta)}\int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N}\left(\frac{\sin\left(\gamma\cdot\frac{s}{2}\right)}{s}\right)^2\cdot(f_A(\beta,\delta)-f_A(\beta,\gamma))^2\,\mathrm{d}s\,\mathrm{d}\mathbf{v}\,\mathrm{d}\beta. \tag{4.17}$$

Using (4.8) and the trivial inequalities $\sin x\geqq\frac{2}{\pi}x$ $\left(0\leqq x\leqq\frac{\pi}{2}\right)$ and $(x-y)^2\geqq$ $\geqq\frac{1}{2}x^2-y^2$ $(x\geqq 0,\ y\geqq 0)$, by (4.17) we have

$$Z_2^+\geqq\left(\frac{\gamma}{\pi}\right)^2\cdot\int_{\Omega_\nu(\delta)}\int_{V(\beta)}\int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N}(f_A(\beta,\delta)-f_A(\beta,\gamma))^2\,\mathrm{d}s\,\mathrm{d}\mathbf{v}\,\mathrm{d}\beta= \tag{4.18}$$

$$=\left(\frac{\gamma}{\pi}\right)^2\cdot\left(\min_{\beta\in\Omega_\nu(\delta)}\mu(V(\beta))\right)\cdot\frac{2}{N}\cdot\left(\int_{\Omega_\nu(\delta)}(f_A(\beta,\delta)-f_A(\beta,\gamma))^2\,\mathrm{d}\beta\right)\mathrm{d}s\,\mathrm{d}\mathbf{v}\geqq$$

$$\geqq\frac{\gamma^2}{10}\cdot\left(\min_{\beta\in\Omega_\nu(\delta)}\mu(V(\beta))\right)\cdot\frac{2}{N}\cdot\left(\frac{1}{2}\int_{\Omega_\nu(\delta)}(f_A(\beta,\delta))^2\,\mathrm{d}\beta-\int_{\Omega_\nu(\delta)}(f_A(\beta,\gamma))^2\,\mathrm{d}\beta\right).$$

By (4.8) and Lemma 4.2A we have (provided $\mu(A)$ is sufficiently large).

$$(4.19)\qquad \int_{\Omega_\nu(\delta)} (f_A(\beta,\gamma))^2\,\mathrm{d}\beta \leqq \int_0^{2\pi} (f_A(\beta,\gamma))^2\,\mathrm{d}\beta \leqq c_{10}\cdot\gamma\cdot d(A).$$

Suppose that

$$(4.20)\qquad \gamma \leqq \frac{(c_{11})^2}{16\cdot c_{12}\cdot\log\left(\frac{1}{\eta}\right)\cdot(c_{10})^2}\cdot\delta.$$

Then from (4.10), (4.18) and (4.19) it follows that

$$Z_2^+ \geqq \frac{\gamma^2}{10}\cdot(2^{\nu-1}\cdot\eta)\cdot\frac{2}{N}\left(\frac{(c_{11})^2}{8\cdot c_{12}\cdot\log\left(\frac{1}{\eta}\right)c_{10}}\cdot\delta\cdot d(A)-c_{10}\cdot\gamma\cdot d(A)\right),$$

that is, by (4.20) we conclude that (provided $\mu(A)$ is sufficiently large)

$$(4.21)\qquad Z_2^+ \geqq c_{13}\cdot 2^\nu\cdot\frac{\eta}{\log\left(\frac{1}{\eta}\right)}\cdot\gamma^2\cdot\delta\cdot\frac{d(A)}{N}.$$

Next we give an upper bound to Z_4^-. By (3.25),

$$I_4=\left[2\pi\cdot\frac{m_0^-}{s}+\frac{\pi}{s},\ M^-+\frac{\pi}{s}+\delta\right)\subset\left[2\pi\cdot\frac{m_0^-}{s}+\frac{\pi}{s},\ 2\pi\cdot\frac{m_0^-}{s}+\frac{3\pi}{2s}\right).$$

Thus, by (4.1) and (4.12) we have

$$(4.22)\quad Z_4^- = \int_{\Omega_\nu(\delta)}\int_{V(\beta)}\int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N}\left(\left|\int_{I_4}\frac{\sin(x\cdot s-\pi)}{s}\cdot\frac{\partial h(x)}{\partial x}\,\mathrm{d}x\right|^+\right)^2\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}\beta \leqq$$

$$\leqq \int_{\Omega_\nu(\delta)}\int_{V(\beta)}\int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N}\left((\max_{x\in I_4}\sin(x\cdot s-\pi))\cdot\frac{1}{s}\right)^2\cdot\left(\int_{I_4}\left|\frac{\partial h(x)}{\partial x}\right|^+dx\right)^2\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}\beta =$$

$$= \int_{\Omega_\nu(\delta)}\int_{V(\beta)}\int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N}\left(\frac{\sin(M^-\cdot s+\delta\cdot s-2\pi m_0^-)}{s}\right)^2\times$$

$$\times\left((\max_{x\in I_4}h(x))-h\left(2\pi\cdot\frac{m_0^-+\frac{1}{2}}{s}\right)\right)^2\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}\beta \leqq \int_{\Omega_\nu(\delta)}\int_{V(\beta)}\int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N}\left(M^-+\delta-\frac{2\pi m_0^-}{s}\right)^2\times$$

$$\times\left((\max_{y\in I_4-M^-}f_A(\beta,y))-f_A\left(\beta,\,2\pi\cdot\frac{m_0^-+\frac{1}{2}}{s}-M^-\right)\right)^2\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}\beta$$

where $I_4-M^-=\left[2\pi\frac{m_0^-+\frac{1}{2}}{s}-M^-,\ \frac{\pi}{s}+\delta\right]$ is the translate of the interval I_4.

Let $u=2\pi\cdot\dfrac{m_0^-+\frac{1}{2}}{s}-M^-$ and $v=\dfrac{\pi}{s}+\delta$. Clearly $[u,v)=I_4-M^-$. From the convexity of A it follows that $\max\limits_{u\leqq y\leqq v} f_A(\beta,y)\leqq\dfrac{v}{u}\cdot f_A(\beta,u)$. Hence

$$(4.23)\qquad \big(\max_{u\leqq y\leqq v} f_A(\beta,y)\big)-f_A(\beta,u)\leqq\frac{v-u}{u}\cdot f_A(\beta,u).$$

We recall (3.21) and (3.24):

$$(4.24)\qquad \left|M^--2\pi\cdot\frac{m_0^-}{s}\right|<\frac{20\eta}{s}\quad\text{and}\quad\frac{40\eta}{s}<\gamma<\delta<\frac{\frac{\pi}{2}-20\eta}{s},$$

thus we have

$$(4.25)\qquad v-u=\delta+M^--2\pi\cdot\frac{m_0^-}{s}<\delta+\frac{20\eta}{s}<2\delta,$$

$$(4.26)\qquad u=\frac{\pi}{s}+2\pi\frac{m_0^-}{S}-M^-\geqq\frac{\pi}{s}-\frac{20\eta}{s}>\frac{\pi}{2s},$$

and so by (4.25) and (4.26),

$$(4.27)\qquad \frac{v-u}{u}\leqq\frac{2\delta}{\frac{\pi}{2s}}=\frac{4\delta}{\frac{\pi}{s}}.$$

We recall: $s\in\left[P(k_0,l_0)-\dfrac{1}{N},\ P(k_0,l_0)+\dfrac{1}{N}\right]$, $\dfrac{\pi}{5\varepsilon_0}\leqq P(k_0,l_0)\leqq\dfrac{2\pi}{5\varepsilon_0}$ and $\varepsilon_0=$ $=(d(A))^{-(1/100)}$. Hence

$$(4.28)\qquad 2\varepsilon_0\leqq\frac{\pi}{\frac{2\pi}{5\varepsilon_0}+1}\leqq\frac{\pi}{P(k_0,l_0)+\frac{1}{N}}\leqq\frac{\pi}{s}\leqq\frac{\pi}{P(k_0,l_0)-\frac{1}{N}}\leqq\frac{\pi}{\frac{\pi}{5\varepsilon_0}-1}\leqq 6\varepsilon_0.$$

Now, by (4.24), (4.26)—(4.28) we obtain that

$$(4.29)\qquad \frac{v-u}{u}\leqq\frac{4\delta}{\frac{\pi}{s}}\leqq\frac{4\delta}{2\varepsilon_0}=\frac{2\delta}{\varepsilon_0}\quad\text{and}$$

$$(4.30)\qquad \varepsilon_0\leqq\frac{\pi}{2s}\leqq u<v=\frac{\pi}{s}+\delta<\frac{3\pi}{2s}\leqq 9\varepsilon_0.$$

From the convexity of A and relation (4.30) it follows that

$$(4.31)\qquad f_A(\beta, u) \leqq \frac{u}{\varepsilon_0}\cdot f_A(\beta, \varepsilon_0) \leqq \frac{9\varepsilon_0}{\varepsilon_0}\cdot f_A(\beta, \varepsilon_0) = 9\cdot f_A(\beta, \varepsilon_0).$$

We return to (4.22). By (4.23), (4.25), (4.29) and (4.31) we have

$$(4.32)\quad Z_4^- \leqq \int_{\Omega_\nu(\delta)} \int_{V(\beta)} \int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N} (v-u)^2\cdot\big((\max_{u\leqq y\leqq v} f_A(\beta, y)) - f_A(\beta, u)\big)^2 \,\mathrm{d}s\,\mathrm{d}v\,d\beta \leqq$$

$$\leqq \int_{\Omega_\nu(\delta)} \int_{V(\beta)} \int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N} (2\delta)^2\cdot\left(\frac{2\delta}{\varepsilon_0}\right)^2\cdot(9\cdot f_A(\beta, \varepsilon_0))^2 \,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}\beta =$$

$$= c_{14}\cdot\frac{\delta^4}{\varepsilon_0^2} \int_{\Omega_\nu(\delta)} \int_{V(\beta)} \int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N} (f_A(\beta, \varepsilon_0))^2 \,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}\beta.$$

We recall:

$$(4.33)\qquad \mu(V(\beta)) < 2^\nu\cdot\eta \quad \text{for all} \quad \beta\in\Omega_\nu(\delta).$$

Moreover, by (4.8) and (4.30), $\varepsilon_0 \leqq \frac{\pi}{2s} \leqq \frac{1}{9\cdot c_9}$, and so by Lemma 4.2A we have,

$$(4.34)\qquad \int_{\Omega_\nu(\delta)} (f_A(\beta, \varepsilon_0))^2 \,\mathrm{d}\beta \leqq \int_0^{2\pi} (f_A(\beta, \varepsilon_0))^2 \,\mathrm{d}\beta \leqq c_{10}\cdot\varepsilon_0\cdot d(A),$$

provided $\mu(A)$ is sufficiently large. Therefore, by (4.32)—(4.34) we have

$$(4.35)\qquad Z_4^- \leqq c_{14}\cdot\frac{\delta^4}{\varepsilon_0^2}\cdot(2^\nu\cdot\eta)\cdot\frac{2}{N}\cdot \int_{\Omega_\nu(\delta)} (f_A(\beta, \varepsilon_0))^2 \,\mathrm{d}\beta \leqq$$

$$\leqq c_{14}\cdot\frac{\delta^4}{\varepsilon_0^2}\cdot(2^\nu\cdot\eta)\cdot\frac{2}{N}\cdot c_{10}\cdot\varepsilon_0\cdot d(A) = c_{15}\cdot 2^\nu\cdot\eta\cdot\frac{\delta^4}{\varepsilon_0}\cdot\frac{d(A)}{N},$$

provided $\mu(A)$ is sufficiently large.

Finally, we give upper bounds to Z_1 and Z_7. By (4.24) we have

$$\max_{M^-\leqq x\leqq M^-+\gamma} \left(\frac{\sin(x\cdot s)}{s}\right)^2 \leqq \left(\frac{\sin(\gamma\cdot s+20\eta)}{s}\right)^2 \leqq \left(\frac{\gamma\cdot s+20\eta}{s}\right)^2 =$$

$$= \left(\gamma+\frac{20\eta}{s}\right)^2 < \left(\frac{3}{2}\gamma\right)^2.$$

Hence by (4.13),

$$(4.36)\qquad Z_1 \leqq \int_{\Omega_\nu(\delta)} \int_{V(\beta)} \int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N} \left(\frac{3}{2}\gamma\right)^2\cdot\left(\int_{M^-}^{M^-+\gamma} \left|\frac{\partial h(x)}{\partial x}\right| \mathrm{d}x\right)^2 \mathrm{d}s\,\mathrm{d}v\,\mathrm{d}\beta.$$

We clearly have

$$\int_{M^-}^{M^-+\gamma}\left|\frac{\partial h(x)}{\partial x}\right|dx=\int_{M^-}^{M^-+\gamma}\left|\frac{\partial h(x)}{\partial x}\right|^+dx-\int_{M^-}^{M^-+\gamma}\left|\frac{\partial h(x)}{\partial x}\right|^-dx,$$

$$\int_{M^-}^{M^-+\gamma}\left|\frac{\partial h(x)}{\partial x}\right|^+dx=\max_{M^-\leqq x\leqq M^-+\gamma}h(x)=\max_{0\leqq y\leqq\gamma}f_A(\beta,y)\quad\text{and}$$

$$-\int_{M^-}^{M^-+\gamma}\left|\frac{\partial h(x)}{\partial x}\right|^-dx\leqq\max_{M^-\leqq x\leqq M^-+\gamma}h(x)=\max_{0\leqq y\leqq\gamma}f_A(\beta,y).$$

Thus, returning to (4.36) we have

$$Z_1\leqq\int_{\Omega_\nu(\delta)}\int_{V(\beta)}\int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N}\left(\frac{3}{2}\gamma\right)^2\cdot(2\max_{0\leqq y\leqq\gamma}f_A(\beta,y))^2\,ds\,d\mathbf{v}\,d\beta. \tag{4.37}$$

We recall:

$$\mu(V(\beta))<2^\nu\cdot\eta\quad\text{for all}\quad\beta\in\Omega_\nu(\delta). \tag{4.38}$$

Moreover, by (4.8) we have

$$\gamma<\min\left\{\frac{1}{9},\frac{1}{9\cdot c_9}\right\}\quad\text{provided }\mu(A)\text{ is sufficiently large.}$$

Hence Lemma 4.2C yields (provided $\mu(A)$ is sufficiently large)

$$\int_{\Omega_\nu(\delta)}(\max_{0\leqq y\leqq\gamma}f_A(\beta,y))^2\,d\beta\leqq\int_0^{2\pi}(\max_{0\leqq y\leqq\gamma}f_A(\beta,y))^2\,d\beta\leqq 4\cdot c_{10}\cdot\gamma\cdot d(A). \tag{4.39}$$

Therefore, by (4.37)—(4.38) we have (provided $\mu(A)$ is sufficiently large)

$$Z_1\leqq(2^\nu\cdot\eta)\cdot\frac{2}{N}\cdot\left(\frac{3}{2}\gamma\right)^2\int_{\Omega_\nu(\delta)}(2\cdot\max_{0\leqq y\leqq\gamma}f_A(\beta,y))^2\,d\beta\leqq \tag{4.40}$$

$$\leqq(2^\nu\cdot\eta)\cdot\frac{2}{N}\cdot\left(\frac{3}{2}\gamma\right)^2\cdot(16\cdot c_{10}\cdot\gamma\cdot d(A))=c_{16}\cdot 2^\nu\cdot\eta\cdot\gamma^3\cdot\frac{d(A)}{N}.$$

Next, by (3.24) and (3.26) we have $\text{length}(I_7)=M^+-2\pi\frac{m_0^+}{s}=M_{\beta,\mathbf{v}}^+-2\pi\frac{m_0^+(\beta,\mathbf{v})}{s}<\frac{14\eta}{s}<\frac{40\eta}{s}<\gamma$. Hence

$$\max_{x\in I_7}\left(\frac{\sin(x\cdot s)}{s}\right)^2\leqq\left(\frac{\sin(\gamma\cdot s)}{s}\right)^2\leqq\gamma^2. \tag{4.41}$$

Moreover, we clearly have

$$\int_{I_7}\left|\frac{\partial h(x)}{\partial x}\right|dx = -\int_{I_7}\left|\frac{\partial h(x)}{\partial x}\right|^{-}dx+\int_{I_7}\left|\frac{\partial h(x)}{\partial x}\right|^{+}dx, \tag{4.42}$$

$$-\int_{I_7}\left|\frac{\partial h(x)}{\partial x}\right|^{-}dx = \max_{x\in I_7} h(x) = \max_{0\leq y\leq \gamma} f_A(\beta+\pi, y) \quad \text{and} \tag{4.43}$$

$$\int_{I_7}\left|\frac{\partial h(x)}{\partial x}\right|^{+}dx \leq \max_{x\in I_7} h(x) = \max_{0\leq y\leq \gamma} f_A(\beta+\pi, y). \tag{4.44}$$

Therefore, by (4.13), (4.41)—(4.44) we obtain that

$$Z_7 \leqq \int_{\Omega_\nu(\delta)}\int_{V(\beta)}\int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N} \gamma^2\cdot\left(\int_{I_7}\left|\frac{\partial h(x)}{\partial x}\right|dx\right)^2 ds\,dv\,d\beta \leqq \tag{4.45}$$

$$\leqq \int_{\Omega_\nu(\delta)}\int_{V(\beta)}\int_{P(k_0,l_0)-1/N}^{P(k_0,l_0)+1/N} \gamma^2\cdot\left(2\max_{0\leqq y\leqq\gamma} f_A(\beta+\pi, y)\right)^2 ds\,dv\,d\beta.$$

We recall:

$$\mu(V(\beta)) < 2^\nu\cdot\eta \quad \text{for all} \quad \beta\in\Omega_\nu(\delta). \tag{4.46}$$

Moreover, by (4.8) we have $\gamma < \min\left\{\frac{1}{9}, \frac{1}{9\cdot c_9}\right\}$ provided $\mu(A)$ is sufficiently large. Hence Lemma 4.2C yields (provided $\mu(A)$ is sufficiently large)

$$\int_{\Omega_\nu(\delta)}\left(\max_{0\leqq y\leqq\gamma} f_A(\beta+\pi, y)\right)^2 d\beta \leqq \int_0^{2\pi}\left(\max_{0\leqq y\leqq\gamma} f_A(\beta+\pi, y)\right)^2 d\beta \leqq 4\cdot c_{10}\cdot\gamma\cdot d(A). \tag{4.47}$$

Therefore, by (4.45)—(4.47) we have (provided $\mu(A)$ is sufficiently large)

$$Z_7 \leqq (2^\nu\cdot\eta)\cdot\frac{2}{N}\cdot\gamma^2\int_{\Omega_\nu(\delta)}\left(2\max_{0\leqq y\leqq\gamma} f_A(\beta+\pi, y)\right)^2 d\beta \leqq \tag{4.48}$$

$$\leqq (2^\nu\cdot\eta)\cdot\frac{2}{N}\cdot\gamma^2\cdot\left(16\cdot c_{10}\cdot\gamma\cdot d(A)\right) = c_{17}\cdot 2^\nu\cdot\eta\cdot\gamma^3\cdot\frac{d(A)}{N}.$$

Now we return to (4.15). From (3.22), (4.20), (4.21), (4.35), (4.40) and (4.48) it follows that

$$\int_0^{2\pi}\int_{R^2}(F_{\tau,\varepsilon_0}(x))^2\,dx\,d\tau \geqq c_8\cdot N^3\left(\frac{1}{4}Z_2^+ - \frac{1}{2}Z_4^- - 2Z_1 - 2Z_7\right) \geqq \tag{4.49}$$

$$\geqq c_8\cdot N^2\cdot d(A)\cdot 2^\nu\cdot\eta\left(\frac{c_{13}\cdot\gamma^2\cdot\delta}{4\cdot\log\left(\frac{1}{\eta}\right)} - \frac{c_{15}\cdot\delta^4}{2\cdot\varepsilon_0} - 2c_{16}\cdot\gamma^3 - 2c_{17}\cdot\gamma^3\right)$$

provided $N \geqq \frac{d(A)+1}{\eta}$, $\gamma \leqq \frac{(c_{11})^2}{16 \cdot c_{12} \cdot \log\left(\frac{1}{\eta}\right) \cdot (c_{10})^2} \cdot \delta$ and $\mu(A) \geqq c_{18}^*$ (c_{18}^* is an "ineffective" constant).

Let $\gamma = \eta^{2/3} \cdot \varepsilon_0$ and $\delta = \eta^{1/2} \cdot \varepsilon_0$ (we recall: $\varepsilon_0 = (d(A))^{-(1/100)}$). A little calculation shows that if η is sufficiently small, say $0 < \eta \leqq c_{19}$, then inequalities (3.24) and (4.20) are satisfied; and we also have,

$$\frac{c_{13} \cdot \gamma^2 \cdot \delta}{4 \cdot \log\left(\frac{1}{\eta}\right)} - \frac{c_{15} \cdot \delta^4}{2\varepsilon_0} - 2c_{16} \cdot \gamma^3 - 2c_{17} \cdot \gamma^3 \geqq \tag{4.50}$$

$$\geqq \frac{c_{13} \cdot \gamma^2 \cdot \delta}{8 \cdot \log\left(\frac{1}{\eta}\right)} = \frac{c_{13}}{8} \cdot \frac{\eta^{11/6}}{\log\left(\frac{1}{\eta}\right)} \cdot (\varepsilon_0)^3.$$

Choosing $\eta = c_{19}$, by (4.49) and (4.50) we obtain that

$$\int_0^{2\pi} \int_{R^2} (F_{\tau,\varepsilon_0}(\mathbf{x}))^2 \, d\mathbf{x} \, d\tau \geqq c_8 \cdot N^2 \cdot d(A) \cdot 2^\nu \cdot \eta \cdot \left(\frac{c_{13}}{8} \cdot \frac{\eta^{11/6}}{\log\left(\frac{1}{\eta}\right)} \cdot (\varepsilon_0)^3 \right) \geqq \tag{4.51}$$

$$\geqq c_{20} \cdot N^2 \cdot d(A) \cdot (\varepsilon_0)^3 = c_{20} \cdot N^2 \cdot (d(A))^{97/100}$$

provided $\mu(A) \geqq c_{18}^*$ and $N \geqq \frac{d(A)+1}{\eta} = \frac{d(A)+1}{c_{19}}$. Note that the hypothesis $\eta = c_{19} \geqq 2 \cdot (d(A))^{-10^{-6}}$ of Lemma 4.1 is also satisfied if $\mu(A) \geqq c_{21}$.

Now we are ready to complete the proof of Theorem 2.1. From formulas (2.2) and (2.3) immediately follows that $|g(\tau A, \mathbf{x}, \varepsilon_0) - \mu(A)| \leqq (d(A))^2$. Therefore, by (3.3), (3.4) and (4.51) we have with $M = \left[N + \frac{1}{2} - d(A) \right]$ (integral part),

$$\int_0^{2\pi} \int_{[-M,M]^2} (g(\tau A, \mathbf{x}, \varepsilon_0) - \mu(A))^2 \, d\mathbf{x} \, d\tau \geqq \tag{4.52}$$

$$\geqq \int_0^{\pi} \int_{R^2} (F_{\tau,\varepsilon_0}(\mathbf{x}))^2 \, d\mathbf{x} \, d\tau - c_{22} \cdot (N \cdot d(A)) \cdot (d(A))^4 \geqq$$

$$\geqq c_{20} \cdot N^2 \cdot (d(A))^{97/100} - c_{22} \cdot N \cdot (d(A))^5.$$

Let $N = c_{23} \cdot (d(A))^5$. If c_{23} is sufficiently large then $N \geqq \frac{d(A)+1}{\eta} = \frac{d(A)+1}{c_{19}}$ and

$$c_{20} \cdot N^2 \cdot (d(A))^{97/100} - c_{22} \cdot N \cdot (d(A))^5 \geqq \frac{c_{20}}{2} \cdot N^2 \cdot (d(A))^{97/100} \tag{4.53}$$

By (4.52) and 4.53) we conclude that (we recall: $\mathcal{U}^2=[0,1)^2$)

$$(4.54)\qquad (2M)^2\cdot\int_0^{2\pi}\int_{\mathcal{U}^2}(g(\tau A,\mathbf{y},\varepsilon_0)-\mu(A))^2\,d\mathbf{y}\,d\tau=$$

$$=\int_0^{2\pi}\int_{[-M,M]^2}(g(\tau A,\mathbf{x},\varepsilon_0)-\mu(A))^2\,d\mathbf{x}\,d\tau\geqq$$

$$\geqq\frac{c_{20}}{2}\cdot N^2\cdot(d(A))^{97/100}>\frac{c_{20}}{8}\cdot(2M)^2\cdot(d(A))^{97/100}.$$

Choosing $c_0=\max\{c_{18}^*, c_{21}\}$ and $c_1=\frac{c_{20}}{16\pi}$, Theorem 2.1 immediately follows from (4.54). It remains to prove Lemmas 4.1—4.2.

5. Proof of Lemma 4.1

Let $p_1=5$, $p_2=13$, $p_3=17, \dots, p_h$ be the first $h=300$ primes in the arithmetic progression $4j+1$, $j=1,2,3,\dots$. By a well known theorem of A. S. Besicovitch [1] the real algebraic numbers $1, \sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \dots, \sqrt{p_h}$ are linearly independent over the rationals. Thus we can apply Schmidt's theorem mentioned at the end of Section 1. Choosing $c=2$, from (1.1) it follows that

$$(5.1)\qquad \|q\cdot\sqrt{p_1}\|\cdot\|q\cdot\sqrt{p_2}\|\dots\|q\cdot\sqrt{p_h}\|\geqq\frac{1}{q^2}$$

for all integers $q\geqq q_0$. Note that q_0 is an "ineffective" threshold constant — we can suppose that q_0 is an integer.

Let $\beta\in[0,2\pi)$ be arbitrary but fixed. We shall show that there are integers $k=k(\beta)$ and $l=l(\beta)$ such that

$$(5.2)\qquad \frac{1}{10\varepsilon_0}\leqq(k^2+l^2)^{1/2}\leqq\frac{1}{5\varepsilon_0}\quad\text{and}\quad\|D_\beta\cdot(k^2+l^2)^{1/2}\|\leqq\eta.$$

(We recall: $D_\beta=M_{\beta,\mathbf{v}}^+-M_{\beta,\mathbf{v}}^-$.) From a classical theorem of Dirichlet it follows that there exist positive integers $n_0\leqq\frac{1}{10\varepsilon_0}$, $n_1\leqq\frac{1}{10\varepsilon_0\sqrt{p_1}}$, $n_2\leqq\frac{1}{10\varepsilon_0\sqrt{p_2}}, \dots, n_h\leqq$ $\leqq\frac{1}{10\varepsilon_0\sqrt{p_h}}$ such that

$$(5.3)\qquad \|n_0\cdot D_\beta\|\leqq 10\varepsilon_0,\ \|n_1\cdot D_\beta\cdot\sqrt{p_1}\|\leqq 10\varepsilon_0\sqrt{p_1},\ \dots,\ \|n_h\cdot D_\beta\cdot\sqrt{p_h}\|\leqq 10\varepsilon_0\sqrt{p_h}.$$

Let $n=\max\{n_0, n_1, n_2, \dots, n_h\}$. We are going to give a lower bound to n. Let $n_0\cdot D_\beta=n^*+\theta$ where n^* is an integer and $|\theta|\leqq 10\varepsilon_0$. Since $n_0\cdot D_\beta\geqq D_\beta\geqq 2r(A)\geqq\frac{2}{9}$

and $\varepsilon_0=(d(A))^{-(1/100)}$, it follows that $n_0\cdot D_\beta\geqq\frac{2}{9}>10\varepsilon_0$ if $\mu(A)$ is sufficiently large. Hence, $n^*\geqq 1$ if $\mu(A)$ is sufficiently large. Let $Q=n^*\cdot\left(\prod_{i=1}^h n_i\right)\cdot q_0$. We have $(1\leqq j\leqq h)$

$$Q\cdot\sqrt{p_j}=(n_0\cdot D_\beta-\theta)\cdot\Big(\prod_{i=1}^h n_i\Big)\cdot q_0\cdot\sqrt{p_j}=$$

$$=\Big(\prod_{\substack{i=1\\ i\neq j}}^h n_i\Big)\cdot q_0\cdot\big(n_j\cdot D_\beta\cdot\sqrt{p_j}\big)-\theta\cdot\Big(\prod_{i=1}^h n_i\Big)\cdot q_0\cdot\sqrt{p_j},\quad\text{and so by (5.3),}$$

$$\|Q\cdot\sqrt{p_j}\|\leqq\Big(\prod_{\substack{i=0\\ i\neq j}}^h n_i\Big)\cdot q_0\cdot\|n_j\cdot D_\beta\cdot\sqrt{p_j}\|+10\varepsilon_0\cdot\Big(\prod_{i=1}^h n_i\Big)\cdot q_0\cdot\sqrt{p_j}\leqq$$

$$\leqq c_{24}\cdot n^h\cdot q_0\cdot\varepsilon_0.$$

Hence

$$\prod_{j=1}^h\|Q\cdot\sqrt{p_j}\|\leqq(c_{24}\cdot n^h\cdot q_0\cdot\varepsilon_0)^h=c_{25}\cdot q_0^h\cdot n^{h^2}\cdot(d(A))^{-h/100}. \tag{5.4}$$

On the other hand, since $Q\geqq q_0$, by (5.1) and 5.4) we have

$$c_{25}\cdot q_0^h\cdot n^{h^2}\cdot(d(A))^{-h/100}\geqq\frac{1}{Q^2}. \tag{5.5}$$

Using the inequality $d(A)\geqq 2r(A)\geqq\frac{2}{9}$, we have

$$Q\leqq(n_0\cdot D_\beta+|\theta|)\cdot\Big(\prod_{j=1}^h n_j\Big)\cdot q_0\leqq n^{h+1}\cdot q_0\cdot d(A)+10\varepsilon_0\cdot n^h\cdot q_0= \tag{5.6}$$

$$=n^{h+1}\cdot q_0\cdot d(A)+10\cdot(d(A))^{-1/100}\cdot n^h\cdot q_0\leqq c_{26}\cdot q_0\cdot n^{h+1}\cdot d(A).$$

Combining (5.5) and (5.6) we obtain that

$$n^{h^2+2h+2}\geqq c_{27}\cdot\frac{(d(A))^{h-200/100}}{q_0^{h+2}} \tag{5.7}$$

If $\mu(A)$ is sufficiently large depending on the value of the threshold constant q_0, then by (5.7) we have (noting that $h=300$)

$$n=\max\{n_0,n_1,\dots,n_h\}\geqq(d(A))^{10^{-5}}. \tag{5.8}$$

Let $n_j=n=\max\{n_0,n_1,\dots,n_h\}$. Then by (5.3) (let $p_0=1$)

$$\|n_j\cdot D_\beta\cdot\sqrt{p_j}\|\leqq 10\varepsilon_0\sqrt{p_j}. \tag{5.9}$$

Let

$$t = \left[\frac{\frac{1}{5\varepsilon_0}}{n_j \sqrt{p_j}} \right] \quad \text{(integral part).}$$

By (5.8) and (5.9) we have

$$(5.10) \qquad \left\| t \cdot n_j \cdot \sqrt{p_j} \cdot D_\beta \right\| \leqq t \cdot 10\varepsilon_0 \sqrt{p_j} \leqq \frac{\frac{1}{5\varepsilon_0}}{n_j \sqrt{p_j}} \cdot 10\varepsilon_0 \sqrt{p_j} =$$

$$= \frac{2}{n_j} = \frac{2}{n} \leqq 2(d(A))^{-10^{-5}} \leqq \eta$$

(in the last step we used the hypothesis of Lemma 4.1). Moreover, using the upper bound $n_j \leqq \frac{1}{10\varepsilon_0 \sqrt{p_j}}$, we have

$$(5.11) \qquad \frac{1}{5\varepsilon_0} \geqq t \cdot n_j \sqrt{p_j} > \left(\frac{\frac{1}{5\varepsilon_0}}{n_j \sqrt{p_j}} - 1 \right) n_j \sqrt{p_j} \geqq \frac{1}{5\varepsilon_0} - \frac{1}{10\varepsilon_0} = \frac{1}{10\varepsilon_0}.$$

Since the prime number p_j satisfies the congruence relation $p_j \equiv 1 \pmod 4$, by a classical theorem of Fermat we obtain the following representation of p_j:

$$p_j = a^2 + b^2, \ a, b \quad \text{integers.}$$

Choosing $k = t \cdot n_j \cdot a$ and $l = t \cdot n_j \cdot b$, (5.2) follows from (5.10) and (5.11).

Now we are ready to complete the proof of Lemma 4.1. Let $k_1 = k_1(\beta)$ and $l_1 = l_1(\beta)$ denote integers satisfying (5.2). We have $M^+_{\beta,\mathbf{v}} = M^+_{\beta,0} + \mathbf{v} \cdot \mathbf{e}(\beta)$ where $\mathbf{e}(\beta) = (\cos\beta, \sin\beta)$. Therefore, a little calculation gives that

$$(5.12) \qquad \mu\left\{\mathbf{v} \in \mathbf{R}^2 \colon |\mathbf{v}| \leqq 1 \quad \text{and} \quad \eta \leqq \{(k_1^2 + l_1^2)^{1/2} \cdot M^+_{\beta,\mathbf{v}}\} \leqq 2\eta\right\} > \eta.$$

Since $D_\beta = M^+_{\beta,\mathbf{v}} - M^-_{\beta,\mathbf{v}}$, from (5.2) it follows that

$$(5.13) \qquad \left\{\mathbf{v} \in \mathbf{R}^2 \colon |\mathbf{v}| \leqq 1 \quad \text{and} \quad \eta \leqq \{(k_1^2 + l_1^2)^{1/2} \cdot M^+_{\beta,\mathbf{v}}\} \leqq 2\eta\right\} =$$

$$= \left\{\mathbf{v} \in \mathbf{R}^2 \colon |\mathbf{v}| \leqq 1, \eta \leqq \{(k_1^2 + l_1^2)^{1/2} \cdot M^+_{\beta,\mathbf{v}}\} \leqq 2\eta \quad \text{and} \right.$$

$$\left. \|(k_1^2 + l_1^2)^{1/2} \cdot M^-_{\beta,\mathbf{v}}\| \leqq 3\eta \right\} \subseteq V(\beta).$$

Lemma 4.1 immediately follows from (5.12) and (5.13). ▮

It remains to prove Lemma 4.2. The forthcoming Part II of this paper will contain the long and technical proof of Lemma 4.2.

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